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Helmholtz Conditions, Covariance, and Invariance Identities

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This paper is concerned with the problem of finding a multiplier matrix g which converts a prescribed system of second-order ordinary differential equations to the Euler-Lagrange form. Sufficient conditions for the existence of a multiplier matrix are given in the form of an infinite system of linear algebraic equations, provided the entries of g may be regarded as components of a $(0, 2)$ symmetric tensor field. As an application, conditions for the local existence of a metric tensor compatible with a given torsion-free connection are deduced.

1. INTRODUCTION

In general, arbitrary systems of second-order ordinary differential equations of the form

$$\ddot{q}^i - f^i(q, \dot{q}, t) = 0, \qquad i = 1, \ldots, s \tag{1}$$

cannot be represented as Euler-Lagrange's equations. However, it is well known that there are cases in which not only one, but rather several inequivalent Lagrangian representations are allowed. This raises a number of problems concerning, e.g., the analysis of the ambiguities in the Lagrangian description and of their connection with the problem of quantization, and the determination of what restrictions are to be imposed on the force term f in order to ensure the existence of at least one Lagrangian function yielding the equations of motion (1) (de Ritis *et al.*, 1983; Marmo *et al.*, 1977; Henneaux, 1982; Santilli, 1978).

The present paper deals with the problem of finding a nonsingular symmetric multiplier matrix $g_{ij}(q, \dot{q}, t)$ such that the representation

$$g_{ij}(\ddot{q}^j - f^j) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \tag{2}$$

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holds in correspondence with a suitable Lagrangian L. The necessary and sufficient conditions for the admissibility of (2) are usually referred to as Helmholtz conditions or self-adjointness conditions; among the equivalent formulations of these equations that are usually found in the literature, the most convenient for our purposes is given by (Santilli, 1978; Crampin, 1981; Sarlet, 1982; Douglas, 1941)

$$\frac{\partial g_{ij}}{\partial \dot{q}^k}=\frac{\partial g_{ik}}{\partial \dot{q}^j}, \qquad g_{ik}\Phi_j^{(0)k}-g_{jk}\Phi_i^{(0)k}=0 \tag{3a,b}$$

$$\frac{dg_{ij}}{dt}+\frac{1}{2}g_{ik}\frac{\partial f^k}{\partial \dot{q}^j}+\frac{1}{2}g_{jk}\frac{\partial f^k}{\partial \dot{q}^i}=0 \tag{3c}$$

where

$$\Phi_j^{(0)k}=\frac{d}{dt}\frac{\partial f^k}{\partial \dot{q}^j}-2\frac{\partial f^k}{\partial q^j}-\frac{1}{2}\frac{\partial f^h}{\partial \dot{q}^j}\frac{\partial f^k}{\partial \dot{q}^h} \tag{4}$$

Helmoltz conditions have already been analyzed in a number of works, mainly concerned with the general theoretical framework of the inverse problem of Lagrangian dynamics, and a rather comprehensive survey of most recent achievements may be found in a paper by Sarlet (1982). On the other hand, several papers have also been devoted to the discussion of the inverse problem in connection with the existense of dynamical symmetries and first integrals of motion (Sarlet et al., 1981; Sarlet et al., 1983; Sarlet, 1983; Schafir, 1981; 1982). This contribution is primarily interested in obtaining sufficiency criteria for the existence of a multiplier matrix, under the assumption that the equations of motion behave covariantly.

To this aim, an infinite sequence of compatibility conditions for (3) is first derived in Section 2. Indeed, these equations are not sufficient to guarantee the existence of a multiplier matrix; however, the requirement of covariance for g gives rise to the additional restrictions to be placed on g in order to allow an efficient discussion of the existence problem.

More specifically, it is shown that the entries g_{ij} transform as the components of a (0, 2) symmetric tensor field under changes of the generalized coordinates iff they depend polynomially on the generalized velocities, with tensor coefficients depending on q and t (Section 3). This condition, in turn, gives rise to a new infinite sequence of linear equations for g_{ij}, which are referred to as invariance identities. Then it is found that algebraic consistency between invariance identities and compatibility conditions guarantees existence of a multiplier matrix (Section 4).

An illustrative application of this result to the theory of gravitation is then exhibited (Section 5): namely, we obtain necessary and sufficient conditions for the local existence of a metric tensor compatible with a given torsion-free linear connection.

2. COMPATIBILITY CONDITIONS

Suppose a generalized force f is given, and consider the associated Helmholtz conditions in the unknown multiplier matrix g. It is the main purpose of this section to derive an infinite sequence of compatibility conditions for the system (3). Besides being of intrinsic interest in themselves, they are also needed to construct the linear system yielding sufficient conditions for the existence of g.

To avoid undue complexity, it is convenient to define by recurrence the following quantities:

$$\Phi_j^{(n+1)k} = \frac{d\Phi_j^{(n)k}}{dt} - \frac{1}{2}\Phi_j^{(n)h}\frac{\partial f^k}{\partial \dot{q}^h} + \frac{1}{2}\Phi_h^{(n)k}\frac{\partial f^h}{\partial \dot{q}^j} \tag{5}$$

where n goes from 0 to infinity and $\Phi_j^{(o)k}$ is given by (4). Then we can state the following result.

Theorem 1. Necessary conditions for equations (3) to admit local analytic solutions are given by

$$\frac{\partial g_{ik}}{\partial q^p}\Phi_j^{(n)k} + g_{ik}\frac{\partial \Phi_j^{(n)k}}{\partial q^p} - (i \leftrightarrow j) = 0 \tag{6a}$$

$$\frac{\partial g_{ik}}{\partial \dot{q}^p}\Phi_j^{(n)k} + g_{ik}\frac{\partial \Phi_j^{(n)k}}{\partial \dot{q}^p} - (i \leftrightarrow j) = 0 \tag{6b}$$

$$g_{ik}\Phi_j^{(n+1)k} - (i \leftrightarrow j) = 0 \tag{6c}$$

$$\frac{\partial g_{ik}}{\partial q^j} + \frac{1}{2}\frac{\partial}{\partial \dot{q}^k}\left(g_{ip}\frac{\partial f^p}{\partial \dot{q}^j}\right) - (i \leftrightarrow j) = 0 \tag{7}$$

where $(i \leftrightarrow j)$ denotes the expression obtained by interchanging i and j in the preceding terms.

Proof. The proof of (6) is given by induction on n. Accordingly, let us consider the case $n = 0$. Then equations (6a) and (6b) are, respectively, the $\partial/\partial q^p$ and $\partial/\partial \dot{q}^p$ derivatives of (3b). Equation (6c) follows from the $\partial/\partial t$ derivative of (3b), which reads

$$\frac{\partial g_{ik}}{\partial t}\Phi_j^{(0)k} + g_{ik}\frac{\partial \Phi_j^{(0)k}}{\partial t} - (i \leftrightarrow j) = 0 \tag{8}$$

In fact, after insertion of the expression for $\partial g_{ij}/\partial t$ obtained from (3c), an application of (6a, b) and (3b) shows that (8) can be rewritten in the equivalent form

$$g_{ik}\left[\frac{d\Phi_j^{(0)k}}{dt}-\frac{1}{2}\Phi_j^{(0)h}\frac{\partial f^k}{\partial \dot{q}^h}+\frac{1}{2}\Phi_h^{(0)k}\frac{\partial f^h}{\partial \dot{q}^j}\right]-(i\leftrightarrow j)=0 \tag{9}$$

In view of the definition (5), equation (9) reduces to (6c) with $n=0$.

Suppose now that (6c) holds with $(n+1)$ replaced by n. Then the proof of (6) is easily obtained by repetition of the procedure which led to the required conditions in the case $n=0$.

In order to prove (7) it is convenient to recall the identity

$$\frac{d}{dt}\frac{\partial h}{\partial \dot{q}^k}=\frac{\partial}{\partial \dot{q}^k}\frac{dh}{dt}-\frac{\partial h}{\partial q^k}-\frac{\partial h}{\partial \dot{q}^p}\frac{\partial f^p}{\partial \dot{q}^k} \tag{10}$$

which holds for any function $h(q, \dot{q}, t)$. Then, substitution of g_{ij} for h into (10) and comparison with (3c) yields

$$\frac{d}{dt}\frac{\partial g_{ij}}{\partial \dot{q}^k}=-\left[\frac{\partial g_{ij}}{\partial q^k}+\frac{\partial g_{ij}}{\partial \dot{q}^p}\frac{\partial f^p}{\partial \dot{q}^k}+\frac{1}{2}\frac{\partial}{\partial \dot{q}^k}\left(g_{ip}\frac{\partial f^p}{\partial \dot{q}^j}+g_{jp}\frac{\partial f^p}{\partial \dot{q}^i}\right)\right] \tag{11}$$

After repeated use of (3a) and relabeling of the indices, equation (11) leads to (7). ∎

Equations (6) and (7) may be used to select a large family of generalized forces f for which the Euler–Lagrange representation (2) is not allowed. In fact, if we recall that every matrix $\Phi_j^{(n)k}$ depends only on the given data and their higher-order derivatives, then we may regard equations (3), (6), and (7) as a linear system that can be solved algebraically for the elements of the multiplier matrix and their derivatives, in correspondence with increasing values for n. Hence, as soon as any incompatibility is found, we conclude that equations (1) cannot be represented as Lagrange's equations.

It is also to be remarked that the linear system under investigation becomes overdetermined in correspondence with very small values of n: for instance, we can already find overdetermination in the case $n=0$, if the dimension of the configuration space is greater than 3. Accordingly, it follows that inner inconsistencies are very likely to appear in connection with arbitrarily chosen data f, and this implies that "in general" there exists no variational formulation for a system of the form (1), in complete agreement with conclusions already drawn by Henneaux (1982) and Henneaux and Shepley (1982).

The previous comments are in close analogy with certain remarks already made by Sarlet (1982). The connection with Sarlet's approach becomes even more transparent when we observe that (6c) and (7) correspond, in a sense, to equations (39k) and (38c) in Sarlet's (1982) paper. However, Sarlet's conditions are derived in a rather different framework:

namely, an investigation of the Cauchy problem for (3) leads to the conclusion that the existence of a multiplier matrix is mathematically equivalent to a proper choice of the initial values for g at the time $t = t_0$, which in turn are characterized as solutions to an overdetermined first-order system involving also an infinite family of algebraic conditions. Additional comments on these points, as well as an alternative derivation of the compatibility conditions, can be found in Caviglia (1984a).

As a final comment, it is worthwhile to observe that, in general, algebraic consistency of the linear system (3), (6), (7) does not seem to be a sufficient condition for the solvability of the self-adjointness condition, so that we should look for the possibility of imposing supplementary restrictions. Indeed, on using Sarlet's reduced formulation (Sarlet, 1982) it has been shown that consistency of (6c) leads to the determination of a multiplier matrix, provided f depends at most linearly on the generalized velocities (Caviglia, 1984b). In the present approach we explore the possibility of introducing natural constraints on the multiplier matrix, rather than on the force term. More precisely, it will be shown in the following sections that, if the g_{ij}s behave covariantly under changes of the generalized coordinates, then a number of linear identities in g_{ij}—the so-called invariance identities—must necessarily be satisfied. The inner consistency of the system formed by equations (3b), (6) and the invariance identities yields the required sufficient conditions for the existence of g.

3. COVARIANCE REQUIREMENT

So far, the inverse problem has been dealt with in full generality. However it appears that, within the context of theoretical mechanics, the expression in the left-hand side of equation (1) may usually be regarded as the ith component of a contravariant vector. On the other hand, it is also well known that the so-called Euler–Lagrange expressions

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k}-\frac{\partial L}{\partial q^k}$$

constitute the components of a covariant vector with respect to changes of the generalized coordinates (Lovelock and Rund, 1975). Therefore it is not surprising that equations (1) have to be multiplied by an integrating matrix, in order to ensure consistency, before any reasonable attempt of finding a variational formulation can be pursued. However, it seems that there has been no systematic analysis of the consequences that can be drawn from the assumption that the multiplier matrix g behaves as a symmetric covariant tensor under changes of the generalized coordinates. Hence, this section is mainly devoted to the determination of the restrictions on the form of g that are imposed by the covariance requirement.

To this aim, let us preliminarly introduce a new family of generalized coordinates $\hat{q}^j$ related to the original coordinates q^k by

$$\hat{q}^j = \hat{q}^j(q^k), \qquad q^k = q^k(\hat{q}^j) \tag{12}$$

The generalized velocities constitute the components of a contravariant vector, that is,

$$\dot{\hat{q}}^j = J^j_k \dot{q}^k \tag{13}$$

where

$$J^j_k = \frac{\partial \hat{q}^j}{\partial q^k}, \qquad K^r_p = \frac{\partial q^r}{\partial \hat{q}^p} \tag{14}$$

Then it turns out that the entries $g_{ij}(q, \dot{q}, t)$ of the multiplier matrix may be regarded as components of a tensor field of type $(0, 2)$ iff the following relation,

$$\hat{g}_{uv}(\hat{q}^h, J^h_i \dot{q}^i, t) = g_{rp}(q^i, \dot{q}^i, t) K^r_u K^p_v \tag{15}$$

holds for any transformation of the form (12), where (14) has also been taken into account. On appealing to techniques that have already been applied to field theories (Lovelock and Rund, 1975; Anderson, 1978; Horndeski, 1981), let us consider the $\partial/\partial\dot{q}^j$ and the $\partial/\partial J^a_b$ derivatives of (15). We obtain, respectively,

$$J^h_j \frac{\partial \hat{g}_{uv}}{\partial \dot{\hat{q}}^h} = \frac{\partial g_{rp}}{\partial \dot{q}^j} K^r_u K^p_v \tag{16}$$

$$\frac{\partial \hat{g}_{uv}}{\partial \dot{\hat{q}}^h} \delta^h_a \delta^b_i \dot{q}^i = g_{rp}(-K^r_a K^b_u K^p_v - K^r_u K^p_a K^b_v) \tag{17}$$

It follows from (16) that the partial derivatives $\partial g_{rj}/\partial \dot{q}^p$ yield the components of a tensor field of type $(0, 3)$. By taking further derivatives with respect to the generalized velocities it may be shown that in general

$$\frac{\partial^p g_{rj}}{\partial \dot{q}^{a_1} \ldots \partial \dot{q}^{a_p}} = g_{rja_1 \ldots a_p} = g_{(rja_1 \ldots a_p)} \tag{18}$$

where round brackets denote symmetrization of the indices enclosed, and $g_{rsa_1 \ldots a_p}$ is a covariant tensor. As a matter of fact, equation (18) implies symmetry with respect to the indices $a_1 \ldots a_p$; however, the condition of total symmetry follows from (3a).

After substitution of the expression for $\partial\hat{g}_{uv}/\partial\dot{\hat{q}}^h$ obtained from (16), (17) yields

$$\frac{\partial g_{uv}}{\partial \dot{q}^a}\dot{q}^b = -g_{av}\delta^b_u - g_{au}\delta^b_v \tag{19}$$

from which it follows that

$$\frac{\partial g_{uv}}{\partial \dot{q}^a}\dot{q}^a = -2g_{uv} \tag{20}$$

On multiplying equation (19) by $\dot{q}^a$ and comparing with (20) it follows that

$$2g_{uv}\dot{q}^b = \dot{q}^a g_{av}\delta^b_u + \dot{q}^a g_{au}\delta^b_v$$

which implies

$$(s-1)g_{ua}\dot{q}^a = 0 \tag{21}$$

The last condition and the arbitrariness of $\dot{q}^a$ contradict the nonsingularity requirement on g if $s \neq 1$; notice that it is not restrictive to assume $s > 1$ because the inverse problem of Lagrangian dynamics in one degree of freedom has been completely solved by Darboux. Therefore, taking also into account (18), it follows that g is allowed to depend on $\dot{q}$ only through a tensorial invariant, in which case equation (17) reduces to an identity. Namely, the most general expression for g_{ij} reads

$$g_{ij} = \sum_{p=0}^{h} g_{ija_1 \ldots a_p}\dot{q}^{a_1} \ldots \dot{q}^{a_p} \tag{22}$$

with $g_{ija_1 \ldots a_p}(q, t)$ totally symmetric covariant tensor. The previous discussion is summarized in the following theorem.

Theorem 2. The multiplier matrix g behaves as a covariant tensor under the transformation (12) iff it is of the form (22).

4. INVARIANCE IDENTITIES AND THE SOLVABILITY OF THE HELMHOLTZ CONDITION

This section aims at exhibiting an infinite sequence of families of invariance identities that may be generated under the assumption that there exists a multiplier matrix given by (22). It is also shown that algebraic consistency of a linear system formed by the invariance identities and the compatibility conditions (6) is sufficient to ensure the existence of a tensorial multiplier matrix. Since the specific details of our analysis to heavily depend on the value of the index h in the expression (22) for g_{ij}, we concentrate our attention on the simplest cases, namely, $h = 0$ and $h = 1$.

Let us consider a multiplier matrix

$$g_{ij} = g_{ij}(q, t) \tag{23}$$

such that equations (3) are identically satisfied. Then the explicit expression of (3c) reads

$$\frac{\partial g_{ij}}{\partial t} + \dot{q}^p \frac{\partial g_{ij}}{\partial q^p} + \frac{1}{2}\left(g_{ik} \frac{\partial f^k}{\partial \dot{q}^j} + g_{jk} \frac{\partial f^k}{\partial \dot{q}^i} \right) = 0 \tag{24}$$

from which it follows that (3c) is mathematically equivalent to

$$\frac{\partial g_{ij}}{\partial q^h} + \frac{1}{2}\left(g_{ik} \frac{\partial^2 f^k}{\partial \dot{q}^h \partial \dot{q}^j} + g_{jk} \frac{\partial^2 f^k}{\partial \dot{q}^h \partial \dot{q}^i} \right) = 0 \tag{25}$$

$$\frac{\partial g_{ij}}{\partial t} + \frac{1}{2} \dot{q}^p \left(g_{ik} \frac{\partial^2 f^k}{\partial \dot{q}^p \partial \dot{q}^j} + g_{jk} \frac{\partial^2 f^k}{\partial \dot{q}^p \partial \dot{q}^i} \right) + \frac{1}{2}\left(g_{ik} \frac{\partial f^k}{\partial \dot{q}^j} + g_{jk} \frac{\partial f^k}{\partial \dot{q}^i} \right) = 0 \tag{26}$$

In fact, (25) is simply the $\partial/\partial \dot{q}^h$ derivative of (24), whereas (26) follows from (24) after substitution of (25). Conversely, multiplication of (25) by $\dot{q}^h$ and addition of (26) yields (24) again.

We are now in a position to describe a step-by-step procedure leading to the determination of the required sequence of sets of integrability conditions that are to be fulfilled by g as a consequence of assumption (23). To find the identities pertaining to the first family F_1, it is convenient to rewrite equations (25) and (26) in the more compact form

$$dg_{ij} + \frac{1}{2}\left(g_{ik} \frac{\partial^2 f^k}{\partial \dot{q}^h \partial \dot{q}^j} + g_{jk} \frac{\partial^2 f^k}{\partial \dot{q}^h \partial \dot{q}^i} \right) (dq^h - \dot{q}^h \, dt) + \frac{1}{2}\left(g_{ik} \frac{\partial f^k}{\partial \dot{q}^j} + g_{jk} \frac{\partial f^k}{\partial \dot{q}^i} \right) dt = 0 \tag{27}$$

so that the integrability conditions are easily obtained by taking the exterior derivative of (27); in view of (25) and (26) their explicit expression reads

$$g_{ik} \frac{\partial^3 f^k}{\partial \dot{q}^p \partial \dot{q}^h \partial \dot{q}^j} + (i \leftrightarrow j) = 0 \tag{28a}$$

$$g_{ik} \left[\frac{\partial^3 f^k}{\partial t \partial \dot{q}^h \partial \dot{q}^j} + \dot{q}^p \left(\frac{\partial^3 f^k}{\partial q^h \partial \dot{q}^p \partial \dot{q}^j} + \frac{1}{2} \frac{\partial^2 f^k}{\partial \dot{q}^p \partial \dot{q}^r} \frac{\partial^2 f^r}{\partial \dot{q}^h \partial \dot{q}^j} - \frac{1}{2} \frac{\partial^2 f^k}{\partial \dot{q}^h \partial \dot{q}^r} \frac{\partial^2 f^r}{\partial \dot{q}^p \partial \dot{q}^j} \right) - \frac{\partial^2 f^k}{\partial q^h \partial \dot{q}^j} - \frac{1}{2} \frac{\partial f^k}{\partial \dot{q}^p} \frac{\partial^2 f^p}{\partial \dot{q}^h \partial \dot{q}^j} + \frac{1}{2} \frac{\partial f^p}{\partial \dot{q}^j} \frac{\partial^2 f^k}{\partial \dot{q}^h \partial \dot{q}^p} \right] + (i \leftrightarrow j) = 0 \tag{28b}$$

$$g_{ik}\left(\frac{\partial^3 f^k}{\partial q^p \partial \dot{q}^h \partial \dot{q}^j}-\frac{\partial^3 f^k}{\partial q^h \partial \dot{q}^p \partial \dot{q}^j}-\frac{1}{2}\frac{\partial^2 f^k}{\partial \dot{q}^p \partial \dot{q}^r}\frac{\partial^2 f^r}{\partial \dot{q}^h \dot{q}^j}\right.$$
$$\left.+\frac{1}{2}\frac{\partial^2 f^k}{\partial \dot{q}^h \partial \dot{q}^r}\frac{\partial^2 f^r}{\partial \dot{q}^p \partial \dot{q}^j}\right)+(i\leftrightarrow j)=0 \tag{28c}$$

Equations (28) constitute the family F_1. Differentiation of every equation of F_1 with respect to each independent variable and substitution from (25) and (26) yields the identities belonging to F_2. Repeated use of this procedure gives rise to an infinite sequence of families of linear equations for the components of g, with coefficients depending on higher-order derivatives of the given data f. These conditions have already been referred to as invariance identities. In so doing we have adopted Lovelock and Rund's (1975) terminology, because equations (28) and their consequences are identically satisfied whenever the system (1) admits a variational formulation and g is a covariant tensor of the form (23).

To complete our analysis of the case $h=0$, we shall now turn to the role played by the invariance identities in the problem of determining whether the system (1) is derivable from a variational principle or not. More specifically, suppose we are looking for a multiplier matrix independent of $\dot{q}$. Then equations (3) are equivalent to the mixed system consisting of the total differential equations (27) and of the finite restrictions (3b). The integrability conditions for the system (27), (3b) are obtained by the same procedure that led to the determination of the invariance identities (Eisenhart, 1961), with the only exception that equations (3b) are now included in the set F_1, from which it follows that F_2 will also contain the derivatives of (3b), and so forth. Accordingly, the integrability conditions are given by the invariance identities plus another suitable sequence of restrictions; it is worth noting that such additional equations can be obtained by substitution into (6) of the proper expressions for the partial derivatives of the components of g.

Denoting by $\mathscr{L}$ the algebraic linear system in the unknowns g_{ij} obtained by the above procedure, we can summarize the previous discussion as follows.

Theorem 3. There exists a multiplier matrix independent of $\dot{q}$ iff the linear system $\mathscr{L}$ is compatible.

As a first practical comment to this result, let us point out that a definite answer as to the compatibility of $\mathscr{L}$ can always be given in a finite number of steps. More specifically, let us consider F_1: if F_1 is not compatible, no solution independent of $\dot{q}$ is allowed. If F_1 is compatible, consider F_2: if F_2 is a consequence of F_1 then the algebraic consistency of $\mathscr{L}$ is ensured;

otherwise, F_2 introduces additional restrictions, and hence the compatibility between F_1 and F_2 is to be discussed following the same procedure already described for F_1. Since there are exactly $S = s(s+1)/2$ unknowns, no more than S steps are required before either compatibility or incompatibility for $\mathscr{L}$ can be proved.

Secondly, we may rephrase our result by stating that algebraic consistency between the compatibility conditions determined in Section 2 and the invariance identities guarantees the existence of a multiplier matrix independent of $\dot{q}$. Of course, it is to be noted here that equation (7) is identically satisfied in view of (25) and (26).

Thirdly, we observe that the condition of independence of g on $\dot{q}$, although highly restrictive from the mathematical viewpoint, is however sufficient to allow the investigation of those classes of holonomic systems that are usually dealt with in the framework of classical mechanics (Arnold, 1978).

Now, let us conclude this section with a few remarks concerning the case $h = 1$. Namely, if we consider a multiplier matrix of the form

$$g_{ij} = \hat{g}_{ij}(q, t) + g_{ijh}(q, t)\dot{q}^h \tag{29}$$

where $g_{ijh} = g_{(ijh)}$, then (3c) can be shown to be equivalent to

$$\frac{\partial \hat{g}_{ij}}{\partial t} = A_{ij} \tag{30a}$$

$$\frac{\partial \hat{g}_{ij}}{\partial q^h} + \frac{\partial g_{ijh}}{\partial t} = B_{ijh} \tag{30b}$$

$$\frac{\partial g_{ijl}}{\partial q^h} + \frac{\partial g_{ijh}}{\partial q^l} = C_{ijhl} \tag{30c}$$

where

$$C_{ijhl} = -\frac{1}{2} g_{ijp} \frac{\partial^2 f^p}{\partial \dot{q}^h \partial \dot{q}^l} - \frac{1}{2}\left(g_{ikl} \frac{\partial^2 f^k}{\partial \dot{q}^j \partial \dot{q}^h} + g_{ikh} \frac{\partial^2 f^k}{\partial \dot{q}^l \partial \dot{q}^j} \right.$$
$$\left. + g_{ik} \frac{\partial^3 f^k}{\partial \dot{q}^h \partial \dot{q}^l \partial \dot{q}^j} \right) + (i \leftrightarrow j) \tag{31a}$$

$$B_{ijh} = -\frac{1}{2} C_{ijhl}\dot{q}^l - \frac{1}{2}\left(g_{ikh} \frac{\partial f^k}{\partial \dot{q}^j} + g_{ik} \frac{\partial^2 f^k}{\partial \dot{q}^h \partial \dot{q}^j} \right) + (i \leftrightarrow j) \tag{31b}$$

$$A_{ij} = -\frac{1}{2}\left(B_{ijh}\dot{q}^h + \frac{1}{4} C_{ijhl}\dot{q}^h \dot{q}^l + g_{ijh} f^h + g_{ik} \frac{\partial f^k}{\partial \dot{q}^j} \right) + (i \leftrightarrow j) \tag{31c}$$

Equations (30) constitute the analog of (25) and (26), but cannot be written as total differentials of the functions $\hat{g}_{ij}$ and g_{ijh}. However, one can introduce auxiliary unknowns $u_{ijh}(q, t)$ and $v_{ijhl}(q, t)$ such that

$$d\hat{g}_{ij} = A_{ij}\,dt + (B_{ijh} - u_{ijh})\,dq^h \tag{32a}$$

$$dg_{ijh} = u_{ijh}\,dt + (C_{ijhl} + v_{ijhl})\,dq^l \tag{32b}$$

Then it may be shown that the integrability conditions for (32) may be written in the form of total differentials for u and v, and hence we are reduced to the situation already examined in the case $h = 0$. Accordingly, we will not proceed further to the investigation of this point, which involves rather long calculations. We only remark that the reduction of the existence problem to the compatibility of a linear system is not peculiar of the case $h = 0$, because similar results also hold when $h = 1$, and may possibly be extended to arbitrary values of h.

5. AN APPLICATION TO THE THEORY OF GRAVITATION

In this section we consider an s-dimensional manifold endowed with a linear torsion-free connection locally described by the so-called connection coefficients $\Gamma^i_{jk} = \Gamma^i_{jk}(q)$. We will derive sufficient conditions for the local existence of a nonsingular metric tensor $g_{ij}(q)$ such that Γ is the Levi-Civita connection of g, i.e., Γ is determined by the condition that parallel transport preserves the scalar product defined by g.

In the four dimensional case, the above problem is related to the analysis of the foundations of the theory of gravitation. For example, according to Trautman's (1966) formulation of the principle of equivalence, the motions of freely falling particles endow space-time with an affine connection which is the only symmetric affine connection whose geodesics coincide with the world lines of free fall. Thus, the characterization of the connections derivable from a metric is strictly related to the determination of allowable trajectories of falling particles in space-time.

The relationship with the inverse problem of the calculus of variations is given by the fact that the metric tensor g is simply introduced into our analysis as a multiplier matrix for the equation of the geodesics of the assigned connection. Namely, we look for a tensor $g_{ij}(q)$ such that the system

$$g_{ij}(\ddot{q}^j + \Gamma^j_{hk}\dot{q}^h\dot{q}^k) = 0 \tag{33}$$

may be expressed in the form of Euler–Lagrange's equations. Furthermore, the condition that g does not depend on $\dot{q}$ allows a determination of g based on the results of Section 4. More specifically, substitution into (25) and (26) of the condition

$$f^k = -\Gamma^k_{jh}\dot{q}^j\dot{q}^h \tag{34}$$

yields

$$\frac{\partial g_{ij}}{\partial q^h} = g_{ik}\Gamma^k_{jh} + g_{jk}\Gamma^k_{ih} \tag{35}$$

$$\frac{\partial g_{ij}}{\partial t} = 0 \tag{36}$$

respectively. Equation (35) can be used to prove that the connection coefficients coincide with the Christoffel symbols of g, whereas (36) shows that g is independent of t, as required.

We now proceed to the construction of the linear system $\mathscr{L}$. When (34) is substituted into the integrability conditions (28), it is found that (28a, b) are identically satisfied; on the contrary, (28c) can be shown to read

$$g_{ik}R^k_{jhp} + g_{jk}R^k_{ihp} = 0 \tag{37}$$

where R^k_{jhp} is the curvature tensor of the given connection. Equation (37) expresses the fact that a curvature tensor which is generated by a metric connection is skew in the first pair of indices; in our approach it is simply the result of the first step in the procedure for the generation of the invariance identities. Since (37) does not depend on $\dot{q}$ and t, in the second step it is only to be differentiated with respect the qs. Then, on comparing with (35) and (37) one finds

$$g_{ik}\nabla_r R^k_{jhp} + g_{jk}\nabla_r R^k_{ihp} = 0 \tag{38}$$

where ∇_r denotes the covariant differentiation operator. By repeated application of the above procedure, we conclude that the invariance identities may be written as

$$g_{ik}\nabla_{r_0}\ldots\nabla_{r_n}R^k_{jhp} + g_{jk}\nabla_{r_0}\ldots\nabla_{r_n}R^k_{ihp} = 0 \tag{39}$$

where the indices r_i go from 0 to infinity.

In order to find the expressions of (3b) and (6), let us observe that (34), (4), and (5) yield

$$\Phi^{(0)k}_j = -2R^k_{hlj}\dot{q}^h\dot{q}^l, \qquad \Phi^{(n)k}_j = -2\nabla_{r_1}\ldots\nabla_{r_n}R^k_{hlj}\dot{q}^{r_1}\ldots\dot{q}^{r_n}\dot{q}^h\dot{q}^l \tag{40a,b}$$

Then (3b) is easily found to be equivalent to

$$g_{ik}R^k_{(hl)j} - g_{jk}R^k_{(hl)i} = 0 \tag{41}$$

Making use of the cyclic identity $R^k_{[hlj]}=0$, which holds because R is the curvature tensor of a symmetric connection, it may be shown that (41) is mathematically equivalent to (37). Accordingly, the equations that may be obtained by differentiation of (41) and which correspond to the family (6), are equivalent to the equations belonging to the set (39). Thus we conclude that algebraic consistency of the linear equations (39) in the unknown components of the metric tensor is a sufficient condition for the existence of a metric. The metric so determined may depend on a certain number of arbitrary constants; these degrees of freedom could possibly be used to satisfy *a priori* requirements on the signature of g.

It seems that the above procedure, aiming at a characterization of linear torsion-free connections compatible with a metric tensor, is simpler than the approach proposed by Cheng and Ni (1980), even though it does not give much insight into the intrinsic geometrical meaning of the integrability conditions, which however has been rather extensively analyzed by Schmidt (1973).

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